# Generation of analytic semigroups by a pair of generalized mixed linear regular ordinary differential operators with interface condition

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**Abstract.** In this paper, we establish with suitable assumptions the analyticity of semigroups generated by a pair of generalized mixed linear regular differential operators

$$\begin{aligned} L_{(n,n)}u(x) &:= \big(L_{1n}u_1(x), L_{2n}u_2(x)\big) \\ &= \Big(\sum_{0\le k\le n} p_k(x)\big(\frac{d}{dx}\big)^k u_1(x), \sum_{0\le k\le n} q_k(x)\big(\frac{d}{dx}\big)^k u_2(x)\Big) \end{aligned}$$

with involving an interface condition in the setting of complex Hilbert space $X = L^2([a,b]) \times L^2([b,c])$. We obtain quite general results that extend previous works by the authors ([3], [10]). The key for showing the generation analytic semigroups will be an inequality of the form

$$Re\langle\big(L_{(n,n)} - \rho I\big)u,\ u\rangle_X + \delta|Im\langle\big(L_{(n,n)} - \rho I\big)u,\ u\rangle_X| \le 0, \forall u \in D(L_{(n,n)})$$

for some constant $\rho > 0$.

**Keywords:** unbounded operators, Dissipative operators, Adjoint, Interface condition, $C_0$- semigroups, analytic semigroups.

## 1. Introduction

Interface problems are a class of problems wherein two different differential equations are defined on two adjacent intervals and the solutions satisfy matching conditions at the point of interface.We encounter these problems in the study of acoustic wave guides [4]. Recently, a new class of problems of the type where different differential operators are defined over two adjacent intervals, involving certain mixed (interface) conditions are studied in [1, 2, 13] and the references therein. These problems involve a pair of differential operators of the type

$$(1.1)\qquad \begin{cases} L_{1n}u_1 = \sum_{0\le k\le n} p_k\big(\frac{d}{dx}\big)^k u_1 = \lambda u_1, \\ \qquad\qquad \text{defined on the interval } I_1 = [a,b] \\ L_{2m}u_2 = \sum_{0\le k\le m} q_k\big(\frac{d}{dx}\big)^k u_2 = \lambda u_2, \\ \qquad\qquad \text{defined on the adjacent interval } I_2 = [b,c], \end{cases}$$

where $-\infty < a < c < +\infty$, $\lambda$ is an unknown constant (eigenvalue) and the functions $u_1$and $u_2$ are required to satisfy certain mixed conditions at the interface $x = b$.

In this paper we consider the following differential system

$$(1.2)\qquad \begin{cases} \frac{du_1}{dt} - \sum_{1\le k\le n} p_k(x)\frac{d^k u_1}{dx^k} = 0 \text{ in } I_1 \\ \frac{du_2}{dt} - \sum_{1\le k\le m} q_k(x)\frac{d^k u_2}{dx^k} = 0 \text{ in } I_2, \end{cases}$$

where $(p_k)_{1\le k\le n}$ (respectively $(q_k)_{1\le k\le m}$ ) are numerical functions on $[a,b]$ (respectively on $[b,c]$). Our goal is to establish some existence results and regularity for solutions of the initial value problem associated with the system (1.2) i.e.,

$$(1.3)\qquad \begin{cases} \frac{du(t)}{dt} - L_{(n,m)}u(t) = 0, \quad t \ge 0 \\ u(0) = u_0 \end{cases}$$

where $L_{(n,m)}$ is the pair of differential operators defined by
(1.4)

$$L_{(n,m)}u = \begin{pmatrix} L_{1n}u_1 \\ L_{2m}u_2 \end{pmatrix} = \begin{cases} L_{1n}u_1 = \sum_{0\le k\le n} p_k\big(\frac{d}{dx}\big)^k u_1 \text{ on } I_1 = [a,b] \\ L_{2m}u_2 = \sum_{0\le k\le m} q_k\big(\frac{d}{dx}\big)^k u_2 \text{ on } I_2 = [b,c]. \end{cases}$$

For $n = m = 2$, a similar works on these problems for regular case have been discussed in [3, 7, 10], and the problem of having singularity at the boundary is discussed in [11].

In [3], T. Bhaskar and R. Kummar have considered the following system on the Hilbert space $X$:

$$(1.5)\qquad \begin{cases} \frac{du_1}{dt} - \tau_1 u_1 = 0 \quad \text{in } I_1, \\ \frac{du_2}{dt} - \tau_2 u_2 = 0 \quad \text{in } I_2, \end{cases}$$

where $\tau_k$ is the second order differential operator $\tau_k u_k = p_k u_k'' + q_k u_k' + r_k u_k$ on $I_k$ for $k = 1, 2$. They proved that under certain assumptions on the matrices $A$ and $B$ that the operator $((\tau_1, \tau_2) - \omega I)$ generates an analytic semigroup of contractions.

In a similar order of ideas, A.Saddi and the present author in [10], proved that certain differential operators of a single variable on an interval $[a, c]$, with matrix coefficients and interface conditions at a point $b \in ]a, c[$, generate analytic semigroups in $L^2$. More precisely, the following second order operator is considered in the space $X = L^2(a, b)^4 \times L^2(b, c)^4$

$$Lu = L\begin{pmatrix} u_{11} \\ u_{12} \\ u_{13} \\ u_{14} \\ u_{21} \\ u_{22} \\ u_{23} \\ u_{24} \end{pmatrix} = \begin{cases} p_{11}u_{11}'' + q_{11}u_{11}' & \\ p_{11}u_{12}'' + q_{11}u_{12}' & \\ p_{12}u_{13}'' + q_{12}u_{13}' & \text{in } [a,\ b] \\ p_{12}u_{14}'' + q_{12}u_{14}' & \\ & \\ p_{21}u_{21}'' + q_{21}u_{21}' & \\ p_{21}u_{22}'' + q_{21}u_{22}' & \\ p_{22}u_{23}'' + q_{22}u_{23}' & \text{in } [b,\ c] \\ p_{22}u_{24}'' + q_{22}u_{24}' & \end{cases}$$

where $p_{ki} \in H^2(a, c)$, $p_{ki}(x) \geq p > 0$ and $q_{ki} \in AC[a, c]$.

It is well know that problem of type (1.3) is well posed in a Banach space $X$ if and only if the operator $(L_{(n,m)}, D(L_{(n,n)}))$ generates a $C_0$-semigroup $(T_t)_{t\geq 0}$ on $X$. Here the solution $u(t)$ is given by $u(t) = T_t u_0$ for the initial data $u_0 \in D(L_{(n,n)})$. For operator semigroups we refer to [5, 8, 9].

The paper is organized as follows: In section 2, we start by setting up our different notions and notations which we shall need in the sequel. In section 3, we study the mixed operator $L_{(n,m)}$ and its adjoint $L^*_{(n,m)}$ and investigate some of its properties. In section 4, we study the $m$-dissipativity of the operator $(L_{(n,n)} - \rho I)$ for some $\rho > 0$. The main results of this section are Lemma 4.1,which is the key to proving our main results, Propositions 4.1 and 4.2 give conditions,when the operator $L_{(n,n)} - \rho I$ and its adjoint are dissipative. Using the results of section 4,we prove in section 5, that under suitable assumptions $L_{(n,n)} - \rho I$ satisfies

$$(1.6)\quad Re\langle \big(L_{(n,n)} - \rho I\big)u,\ ; u\rangle_X + \delta |Im\langle \big(L_{(n,n)} - \rho I\big)u\ u\rangle_X| \leq 0 \quad \forall\, u \in D(L_{(n,n)})$$

with $\rho > 0$, so that $L$ generates an analytic semigroup on $X$ (see [6]).

## 2. Functional setting of the problem

We shall introduce a few notations and make some assumptions. Let $\mathbb{R}$ and $\mathbb{C}$ represent the real and complex number fields and let $M_n(\mathbb{K})$, $(K = \mathbb{R}$, or $\mathbb{C})$ be the space of square $n \times n$ matrix with coefficients in $\mathbb{K}$. For any compact interval $I$ of $\mathbb{R}$ and for a nonnegative integer $k$, let $C^k(I,\mathbb{C})$ denote the space of all $k$-times continuously differentiable complex valued functions defined on $I$. Let $AC^k(I,\mathbb{C})$ denote the space of all complex valued functions $\varphi$ which have $(k-1)$ continuous derivatives in $I$ and the $(k-1)^{th}$ derivative of $\varphi$ absolutely continuous in $I$. For a function $\varphi$, let $\varphi^{(j)}$ denote the $j^{th}$ derivative of $\varphi$, if it exists. If $\varphi$ is any function with $(n-1)$-derivatives, the vector $k[\varphi] = column(\varphi, \varphi', ..., \varphi^{(n-1)})$ is called the Wronskian of $\varphi$. For any $n\times m$ matrix $A$, let $A^*$ denote the adjoint of $A$. For a square matrix $A$, $A^{-1}$ denotes the inverse of $A$, if it exists. For any two nonempty sets(topological spaces) $X_1$ and $X_2$, let $X_1 \times X_2$ denote the cartesian product (space equipped with product topology) of $X_1$ and $X_2$, taken in that order.

Let $I_1 = [a,\ b]$ and $I_2 = [b,\ c],$ where $-\infty < a < b < c < +\infty$. We introduce the functions spaces $E(I_k,\mathbb{C}) = \{\varphi : I_k \to \mathbb{K}/\ \varphi\ \text{ measurable }\}$ and denote by $L^2(I_k,\ \mathbb{C})$ the Hilbert space defined by

$$L^2(I_k,\mathbb{C}) = \Big\{\varphi \in E(I_k,\ \mathbb{C})/ \int_{I_k} |\varphi(x)|^2 dx < +\infty\Big\},$$

endowed with the inner product $\langle\ .\ \rangle_{L^2(I_k,\ \mathbb{C})}$ and the norm $\|\ .\ \|_{L^2(I_k,\ \mathbb{C})}$ given by

$$\langle\varphi\ ,\psi\rangle_{L^2(I_k,\ \mathbb{C})} = \int_{I_k} (\varphi\overline{\psi})(x)dx, \quad \text{for } k = 1,\ 2$$

and

$$\|\varphi\|_{L^2(I_k,\ \mathbb{C})} = \left(\int_{I_k} |\varphi(x)|^2 dx\right)^{\frac{1}{2}} \quad \text{for } k = 1,\ 2.$$

We consider the product Hilbert space $X = L^2(I_1,\ \mathbb{C}) \times L^2(I_2,\ \mathbb{C})$ . Notice $X$ is a Hilbert space with the scalar product $\langle\ .\ \rangle_X$ and the norm $\|\ .\ \|_X$ given by

$$\langle u,\ v\ \rangle_X = \langle u_1,\ v_1\rangle_{L^2(I_1,\ \mathbb{C})} + \langle u_2,\ v_2\rangle_{L^2(I_2,\ \mathbb{C})} \tag{2.1}$$

for all $u = (u_1,\ u_2),\ v = (v_1,\ v_2) \in X$ and

$$\|(u_1\ ,\ u_2)\|_X = \left(\ \|u_1\|^2_{L^2(I_1,\mathbb{C})} + \|u_2\|^2_{L^2(I_2,\mathbb{C})}\right)^{\frac{1}{2}},\ (u_1,u_2) \in X.$$

We consider the following spaces,

$$\mathcal{H}^n(I_1) = \Big\{\varphi \in AC^m(I_1,\ \mathbb{C})\ \text{ suth that }\ \varphi\ \text{ and }\ \varphi^{(n)}\ \text{ are both in }\ L^2(I_k,\ \mathbb{C})\Big\}$$

and

$$\mathcal{H}^m(I_2) = \Big\{\varphi \in AC^m(I_2,\ \mathbb{C})\ \text{ suth that }\ \varphi\ \text{ and }\ \varphi^{(m)}\ \text{ are both in }\ L^2(I_2,\ \mathbb{C})\Big\}.$$

Let $\mathcal{H}(I_1 \times I_2) = \mathcal{H}^n(I_1) \times \mathcal{H}^m(I_2)$ be the cartesian product Hilbert space with the inner product

$$\begin{aligned} \langle u,\ v\rangle_{\mathcal{H}} &= \langle (u_1, u_2)\ , (v_1, v_2)\rangle_{\mathcal{H}} \\ &= \sum_{0\le k\le n} \int_{I_1} u_1^{(k)} \overline{v_1}^{(k)} dx + \sum_{0\le k\le m} \int_{I_2} u_2^{(k)} \overline{v_2}^{(k)} dx \end{aligned}$$

and the associated norm

$$\|u\|_{\mathcal{H}} = \Big( \sum_{0\le k\le n} \int_{I_1} |u_1^{(k)}|^2 dx + \sum_{0\le k\le m} \int_{I_2} |u_2^{(k)}|^2 dx \Big)^{\frac{1}{2}}.$$

For $u_1 \in \mathcal{H}^n(I_1)$, and $u_2 \in \mathcal{H}^m(I_2)$, we denote

$$k[u_1](b) = Column\Big(u_1(b), u_1'(b), u_1^{(2)}(b)..., u_1^{(n-1)}(b)\Big) \tag{2.2}$$

and

$$k[u_2](b) = Column\Big(u_2(b), u_2'(b), u_2^{(2)}(b)..., u_2^{(m-1)}(b)\Big). \tag{2.3}$$

Let $A \in M_n(\mathbb{C})$ and $B \in M_m(\mathbb{C})$ be non singular matrices with complex entries.

Let $L_{1n}$ and $L_{2m}$ be a pair of formal ordinary differential operators of order $n$ and $m$ defined on the intervals $I_1$ and $I_2$, respectively, of the form

$$L_{1n}u_1 = \sum_{0\le k\le n} p_k \big(\frac{d}{dx}\big)^k u_1 = \sum_{0\le k\le n} p_k D^k u_1 \ \text{ on } \ I_1$$

and

$$L_{2m}u_2 = \sum_{0\le k\le m} q_k \big(\frac{d}{dx}\big)^k u_2 = \sum_{0\le k\le m} q_k D^k u_2 \ \text{ on } \ I_2,$$

We make the following assumptions

$$(H_1) \begin{cases} p_n \in \mathcal{H}^n(I_1),\ p_n > 0 \ \text{ on } I_1 \\ p_k \in AC^k(I_1), \ \ k = 1, 2, ....n-1 \\ p_2 > 0 \ \text{ and } \ p_2' > 0 \ \text{ on } I_1 \\ p_0 \ \text{ piecewise continuous on } \ I_1. \end{cases}$$

$$(H_2) \begin{cases} q_m \in \mathcal{H}^m(I_2),\ q_m > 0 \ \text{ on } I_2 \\ q_k \in AC^k(I_2), \ \ k = 1, 2, ....m-1 \\ q_2 > 0 \ \text{ and } \ q_2' > 0 \ \text{ on } I_2 \\ q_0 \ \text{ piecewise continuous on } \ I_2. \end{cases}$$

and the interface condition at the point x = b, for $(u_1, u_2) \in \mathcal{H}(I_1 \times I_2)$

$$(H_3) \qquad Ak[u_1](b) = Bk[u_2](b).$$

**Remark 2.1.** If $\varphi$ and $\psi \in C^k(I_1)$ $\big($or $\varphi,\ \psi \in C^k(I_2)\big)$, then the following formula is easily verified

$$\psi\varphi^{(k)} = (-1)^k\varphi\psi^{(k)} + \frac{d}{dt}\bigg(\sum_{0\le j\le k-1}(-1)^j\psi^{(j)}\varphi^{(k-1-j)}\bigg) \tag{2.4}$$

## 3. Mixed operator $(L_{(n,m)}, D(L_{(n,m)}))$ and its adjoint

In order to study the operator $(L_{(n,m)}, D(L_{(n,m)}))$, we introduce its Green formula. We will be able to obtain some characteristic proprieties.

According to [12] the corresponding formal Lagrange adjoint expressions of $L_{1n}$ and $L_{2m}$ are given as

$$L^*_{1n}u_1 = \sum_{0\le k\le n}(-1)^k\bigg(\frac{d}{dx}\bigg)^k p_k u_1 \ \text{ and } \ L^*_{2m}u_2 = \sum_{0\le k\le m}(-1)^k\bigg(\frac{d}{dx}\bigg)^k q_k u_2.$$

Consider now the operator $(L_{(n,m)},\ D(L_{(n,m)}))$ given by

$$\begin{cases} D(L_{(n,m)}) = \Big\{(u_1,\ u_2) \in \mathcal{H}(I_1\times I_2)/Ak[u_1](b) = Bk[u_2](b), \\ \qquad\qquad\qquad \left.\begin{array}{l}\beta_a^k = 0, k = 1,2,....n-1 \\ \beta_c^j = 0, j = 1,2,...,m-1\end{array}\right\} \\ L_{(n,m)}u = \big(L_{1n}u_1, L_{2m}u_2\big) \\ = \Big(\displaystyle\sum_{0\le k\le n} p_k\big(\frac{d}{dx}\big)^k u_1,\ \sum_{0\le k\le m} q_k\big(\frac{d}{dx}\big)^k u_2\Big) \end{cases} \tag{3.1}$$

with $\beta_a^k = u_1^{(k)}(a) - \gamma_a^k u_1(a)$ for $k = 1,2,...,n-2$ and $\beta_a^{n-1} = u_1^{(n-1)}(a)$ and $\beta_c^j = u_2^{(j)}(c) - \delta_c^j u_2(c)$ for $j = 1,2,...,m-2$ and $\gamma_c^{m-1} = u_2^{(m-1)}(c)$ $\gamma_a^k$ and $\delta_c^j$ are here fixed real numbers.

**Theorem 3.1.** *The operator* $(L_{(n,m)},\ D(L_{(n,m)}))$ *is a densely defined closed unbounded linear operator in* $X$.

**Proof.** According to [14], the operators $(L_{1n},\ \mathcal{H}^n(I_1))$ (resp. $(L_{2m}, \mathcal{H}^m(I_2))$) given in (3.1) is densely defined and closed on $L^2(I_1,\ \mathbb{C})$ (resp. $L^2(I_2,\ \mathbb{C})$)). Hence the result (see also [15], Theorem 3.6). $\square$

**Green's formula for $L_{(n,m)}$:** Let $u = (u_1,\ u_2) \in D(L_{(n,m)})$ and $v = (v_1,\ v_2) \in \mathcal{H}$, the Green's formula takes the form

$$\begin{aligned}\big\langle L_{(n,m)}u, v\big\rangle_X &= \big\langle L_{(n,m)}(u_1,u_2),(v_1,v_2)\big\rangle_X \\ &= \sum_{1\le k\le m}\ \sum_{0\le j\le k-1}(-1)^j(q_k\overline{v_2})^{(j)}u_2^{(k-1-j)}(c)\end{aligned}$$

$$
\begin{aligned}
&- \sum_{1\le k\le n}\sum_{0\le r\le k-1}(-1)^n(p_k\overline{v_1})^{(r)}u_1^{(k-1-r)}(a)\\
&+\Big\{\sum_{1\le k\le m}\sum_{0\le j\le k-1}(-1)^j(q_k\overline{v_2})^{(j)}u_2^{(k-1-j)}\\
&-\sum_{1\le k\le n}\sum_{0\le r\le k-1}(-1)^r(p_k\overline{v_1})^{(r)}u_1^{(k-1-r)}\Big\}(b)\\
&+\int_a^b\overline{(L_n^*v_1)}u_1(x)dx+\int_b^c\overline{(L_m^*v_2)}u_2(x)dx\\
&=\sum_{1\le k\le m}\Lambda_{ck}^*u_2^{(k-1-j)}(c)-\sum_{1\le k\le n}\Lambda_{ak}^*u_1^{(k-1-r)}(a)\\
&+\Big\{\sum_{1\le k\le m}\sum_{0\le j\le k-1}(-1)^j(q_k\overline{v_2})^{(j)}u_2^{(k-1-j)}\\
&-\sum_{1\le k\le n}\sum_{0\le r\le k-1}(-1)^r(p_k\overline{v_1})^{(r)}u_1^{(k-1-r)}\Big\}(b)\\
&+\int_a^b\overline{(L_n^*v_1)}u_1(x)dx+\int_b^c\overline{(L_m^*v_2)}u_2(x)dx
\end{aligned}
$$

where

$$\Lambda_c^{*k}=\sum_{0\le j\le k-1}(-1)^j\sum_{0\le i\le j}\binom{j}{i}q_k^{(j-i)}(c)\overline{v_2}^{(i)}(c),\quad k=1,2,...,m$$

and

$$\Lambda_a^{*k}=\sum_{0\le r\le k-1}(-1)^r\sum_{0\le i\le r}\binom{r}{i}p_k^{(r-i)}(a)\overline{v_1}^{(i)}(a),\quad k=1,2,...,n.$$

Let us define two matrix $P=(p_{ij})_{1\le i,j\le n}\in M_n(\mathbb{R})$ and $Q=(q_{ij})_{1\le i,j\le n}\in M_m(\mathbb{R})$ as follows

$$P=(p_{ij})_{1\le i,j\le n}=\left(\sum_{1\le k,r\le n-1}a_{ij}^rp_r^{(k)}(b)\right)_{1\le i,j\le n},\quad a_{ij}^r\in\mathbb{R}$$

and

$$Q=\big(q_{ij}\big)_{1\le i,\,j\le m}=\left(\sum_{1\le k,r\le m-1}c_{ij}^rq_r^{(k)}(b)\right)_{1\le i,j\le m},\quad c_{ij}^r\in\mathbb{R}.$$

Now, by using the matrix $P$ and $Q$, we verify that

$$
\begin{aligned}
&\sum_{1\le k\le m}\sum_{0\le j\le k-1}(-1)^j(q_k\overline{v_2})^{(j)}u_2^{(k-1-j)}(b)\\
&-\sum_{1\le k\le n}\sum_{0\le r\le k-1}(-1)^r(p_k\overline{v_1})^{(r)}u_1^{(k-1-r)}(b)
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\overline{v_2(b)}, \overline{v_2'(b)}, ..., \overline{v_2^{(m-1)}(b)}\right)\left(\sum_{1\le k,r\le m-1} c_{ij}^r q_r^{(k)}(b)\right)_{1\le i,j\le m} \\
&\quad \cdot \left(u_2(b), u_2'(b), ..., u_2^{(m-1)}(b)\right) \\
&- \left(\overline{v_1(b)}, \overline{v_1'(b)}, ..., \overline{v_1^{(n-1)}(b)}\right)\left(\sum_{1\le k,r\le n-1} a_{ij}^r p_r^{(k)}(b)\right)_{1\le i,j\le n} \\
&\quad \cdot \left(u_1(b), u_1'(b), ..., u_1^{(n-1)}(b)\right))
\end{aligned}
$$

From (2.2) and (2.3) and the fact that $(u_1, u_2) \in D(L_{(n,m)})$, it follows that,

$$
\begin{aligned}
\langle L_{(n,m)}u,\ v\rangle_X = &\sum_{1\le k\le m} \Lambda_c^{*k} u_2^{(k-1-j)}(c) \\
&- \sum_{1\le k\le n} \Lambda_a^{*k} u_1^{(k-1-r)}(a) + \left(\overline{k[v_2]}Qk[u_2] - \overline{k[v_1]}Pk[u_1]\right)(b) \\
&+ \int_a^b \overline{(L_{1n}^* v_1)} u_1(x)dx + \int_b^c \overline{(L_{2m}^* v_2)} u_2(x)dx.
\end{aligned}
$$

Introducing the new matrix

$$
M_1 = (A^{-1})^* P^* \text{ and } M_2 = (B^{-1})^* Q^*
$$

and using the interface condition $(H_3)$ ,we then transform the Green's formula to

$$
\begin{aligned}
&\langle L_{(n,m)}u,\ v\rangle_{\mathcal{H}} = \sum_{1\le k\le m} \Lambda_c^{*k} u_2^{(k-1-j)}(c) - \sum_{1\le k\le n} \Lambda_a^{*k} u_1^{(k-1-r)}(a) \\
&+ \left((M_2 k[v_2])^* Bk[u_2] - (M_1 k[v_1])^* Ak[u_1]\right)(b) + \langle u_1,\ L_{1n}^* v_1\rangle + \langle u_2,\ L_{2m}^* v_2\rangle.
\end{aligned}
$$

The following proposition characterize the adjoint operator of $L_{(n,m)}$.

**Proposition 3.1.** *Let $(L_{(n,m)},\ D(L_{(n,m)}))$ be the operator given as in* (3.1). *Then its adjoint $(L_{(n,m)}^*,\ D(L_{(n,m)}^*))$ is a densely defined closed unbounded operator defined by*

$$
(3.2)\begin{cases}
D(L_{(n,m)}^*) = \Big\{(v_1,\ v_2) \in \mathcal{H}(I_1 \times I_2)/\ M_1 k[v_1](b) = M_2 k[v_2](b), \\
\qquad\qquad \beta_a^{*k} = 0, k = 1, 2, ....n-1 \\
\qquad\qquad \beta_c^* j = 0, j = 1, 2, ..., m-1 \quad \Big\} \\
L_{(n,m)}^*(v_1,\ v_2) = (L_{1n}^* v_1,\ L_{2m}^* v_2).
\end{cases}
$$

*where $\beta_a^{*k} = v_1^{(k)}(a) - \omega_a^k v_1(a),\ k = 1, 2, ..., n-1$ and $\beta_c^{*j} = v_2^{(j)}(c) - \omega_c^j v_2(c),\ j = 1, 2, ..., m-1$ $\omega_a^k$ and $\omega_c^j$ are here fixed real numbers.*

**Proof.** Let $T$ be the linear operator defined by

$$\begin{cases} D(T) = \Big\{(v_1,\ v_2) \in \mathcal{H}/\ M_1k[v_1](b) = M_2k[v_2](b), \\ \qquad\qquad \left.\begin{array}{l}\beta_a^{*k} = 0, k = 1, 2, \ldots .n-1 \\ \beta_c^{*j} = 0, j = 1, 2, \ldots, m-1\end{array}\right\} \\ T(v_1,\ v_2) = (L_{1n}^* v_1,\ L_{2m}^* v_2) \end{cases}$$

We have to show that $L_{(n,m)}^* = T$. From the Green's formula, it follows that $D(T) \subset D(L_{(n,m)}^*)$. Now we need to prove the following equality

$$\langle L_{(n,m)}(u_1,\ u_2),\ (v_1,\ v_2)\rangle = \langle (u_1,\ u_2),\ L_{(n,m)}^*(v_1,\ v_2)\rangle,$$

for all $(u_1,\ u_2) \in D(L_{(n,m)})$ and $(v_1,\ v_2) \in D(L_{(n,m)}^*)$, which follows if we prove that

$$\begin{aligned} &\sum_{1\le k\le m} \Lambda_c^{*k} u_2^{(k-1-j)}(c) - \sum_{1\le k\le n} \Lambda_a^{*k} u_1^{(k-1-r)}(a)) \\ &+ \big((M_2k[v_2])^* BK[u_2] - (M_1k[v_1])^* Ak[u_1]\big)(b)) = 0. \end{aligned}$$

Choose $(u_1,\ u_2) \in D(L_{(n,m)})$ verifying $u_1(a) = u_2(c) = 0$; then we obtain

$$\big((M_2k[v_2])^* BK[u_2] - (M_1k[v_1])^* Ak[u_1]\big)(b)) = 0,$$

Green's formula implies that

$$\sum_{1\le k\le m} \Lambda_c^{*k} u_2^{(k-1-j)}(c) - \sum_{1\le k\le n} \Lambda_a^{*k} u_1^{(k-1-r)}(a) = 0.$$

An appropriate choice of functions $(u_1,\ u_2) \in D(L_{(n,m)})$, we get $\beta_a^{*k} = \beta_c^{*j} = 0, 1 \le k \le n-1, 1 \le j \le m-1$ and $M_1k[v_1](b) = M_2k[v_2](b)$. This implies that $D(L_{(n,m)}^*) \subset D(T)$ and therefore $L_{(n,m)}^* = T$ as required. $\square$

**Lemma 3.1.** *For $v = (v_1, v_2) \in D(L_{(n,m)}^*)$ we have*

$$\begin{aligned} \langle L_{(n,m)}^* v,\ v\rangle_X = &\sum_{0\le k\le n} (-1)^k \int_a^b p_k v_1 (\overline{v_1})^{(k)} dx \\ &+ \sum_{1\le k\le n} \sum_{0\le j\le k-1} (-1)^j (\overline{v_1})^{(j)} (p_k v_1)^{(k-1-j)} \Big]_a^b \\ &+ \sum_{0\le k\le m} (-1)^k \int_b^c q_k v_2 (\overline{v_2})^{(k)} dx \\ &+ \sum_{1\le k\le m} \sum_{0\le j\le k-1} (-1)^j (\overline{v_2})^{(j)} (q_k v_2)^{(k-1-j)} \Big]_b^c . \end{aligned}$$

**Proof.** From (2.1) we have

$$\begin{aligned}\langle L^*_{(n,m)}v,\ v\rangle_X &= \langle L^*_n v_1,\ v_1\rangle + \langle L^*_m v_2,\ v_2\rangle \\ &= \underbrace{\sum_{0\le k\le n}(-1)^k\int_a^b (p_k v_1)^{(k)}\overline{v_1}(x)dx}_{I} \\ &\quad + \underbrace{\sum_{0\le k\le m}(-1)^k\int_b^c (q_k v_2)^{(k)}\overline{v_2}(x)dx}_{J}\end{aligned}$$

Let us examine the term $I$,using the identity (2.4) we get for $k\ge 1$

$$\begin{aligned}\overline{v_1}(p_k v_1)^{(k)} &= (-1)^k p_k v_1(\overline{v_1})^{(k)} \\ &\quad + \frac{d}{dx}\Big(\sum_{0\le j\le k-1}(-1)^j(\overline{v_1})^{(j)}(p_k v_1)^{(k-1-j)}\Big).\end{aligned}$$

Therefore, we deduce that

$$\begin{aligned}\int_a^b (p_k v_1)^{(k)}\overline{v_1}(x)dx &= (-1)^k\int_a^b p_k v_1(\overline{v_1})^{(k)}(x)dx \\ &\quad + \sum_{0\le j\le k-1}(-1)^j(\overline{v_1})^{(j)}(p_k v_1)^{(k-1-j)}\Big]_a^b.\end{aligned}$$

A similar argument applies to the term $J$ gives

$$\begin{aligned}\int_b^c (q_k v_2)^{(k)}\overline{v_2}(x)dx &= (-1)^k\int_b^c q_k v_2(\overline{v_2})^{(k)}(x)dx \\ &\quad + \sum_{0\le j\le k-1}(-1)^j(\overline{v_2})^{(j)}(q_k v_2)^{(k-1-j)}\Big]_b^c.\end{aligned}$$

□

In the rest of the work, we assume that $n = m \ge 3$.

## 4. $m$- Dissipativity of the operator $(L_{(n,n)}, D(L_{(n,n)}))$.

In this section we prove the $m$-dissipativity of the operator $(L_{(n,n}, D(L_{(n,n)}))$.

**Definition 4.1** ([9]). *A linear closed densely defined operator $(T,\ D(T))$ on a complex Hilbert space $X$ is called dissipative if*

$$\forall\ u \in D(T) \subset X,\ \ Re\langle Tu,\ u\rangle \le 0.$$

**Definition 4.2** ([9]). *A dissipative operator $(T,\ D(T))$ on a Hilbert space $X$ is called m-dissipative if there exists $\lambda > 0$ such that $\mathcal{R}(\lambda I - T) = X$.*

We establish the following technical lemma which is the key to proving our main results.

**Lemma 4.1.** *If* $u \in C^k([a,\ b],\ \mathbb{R})$ *and* $v \in C^k([a,\ b],\ \mathbb{C})$ *then the following properties hold.*

1. *If* $k = 1$ *then*

$$2Re \int_a^b (u\overline{v}v')(x)dx = \{u(b)|v|^2(b) - u(a)|v|^2(a)\} - \int_a^b u'(x)|v|^2(x)dx$$

2. *If* $k \geq 2$, *then*

$$\begin{aligned}2Re \int_a^b (u\overline{v}v^{(k)})(x)dx &= (-1)^k \int_a^b (u^{(k)}|v|^2)(x)dx \\ &+ \sum_{0\leq j\leq k-1} \left[u^{(j)}(|v|^2)^{(k-1-j)}(b) - u^{(j)}(|v|^2)^{(k-1-j)}(a)\right] \\ &- \sum_{1\leq j\leq k-1} \binom{k}{j} \int_a^b uv^{(j)}\overline{v}^{(k-j)}(x)dx.\end{aligned}$$

**Proof.** 1. Let $Z = \displaystyle\int_a^b (u\overline{v}v')(x)dx$, we have

$$2Re(Z) = \int_a^b \big(u(\overline{v}v' + \overline{v'}v)(x)dx = \int_a^b \big(u(|v|^2)'\big)(x)dx$$

hence 1. follows.

2. It is will know that

$$2Re \int_a^b (u\overline{v}v^{(k)})(x)dx = \int_a^b u\big(v\overline{v}^{(k)} + v^{(k)}\overline{v}\big)(x)dx.$$

On the other hand the classical Leibnitz formula gives

$$\big(v\overline{v}\big)^{(k)} = \sum_{0\leq j\leq k} \binom{k}{j} v^{(j)}(\overline{v})^{(k-j)} = v(\overline{v})^{(k)} + \sum_{1\leq j\leq k-1} \binom{k}{j} v^{(j)}(\overline{v})^{(k-j)} + v^{(k)}\overline{v}.$$

Hence

$$\begin{aligned}&u\big(v\overline{v}^{(k)} + v^{(k)}\overline{v}\big) = u(|v|^2)^{(k)} - \sum_{1\leq j\leq k-1} \binom{k}{j} uv^{(j)}(\overline{v})^{(k-j)} \\ &= (-1)^k u^{(k)}|v|^2 + \frac{d}{dt} \sum_{0\leq j\leq k-1} (-1)^j u^{(j)}(|v|^2)^{(k-1-j)} - \sum_{1\leq j\leq k-1} \binom{k}{j} uv^{(j)}(\overline{v})^{(k-j)}.\end{aligned}$$

It follows that

$$\int_a^b u\big(v\overline{v}^{(k)} + v^{(k)}\overline{v}\big)(x)dx = (-1)^k \int_a^b u^{(k)}|v|^2(x)dx$$
$$+ \sum_{0\le j\le k-1} (-1)^j \big[u^{(j)}(|v|^2)^{(k-1-j)}(x)\big]_a^b - \sum_{1\le j\le k-1} \binom{k}{j} \int_a^b uv^{(j)}(\overline{v})^{(k-j)}(x)dx.$$

□

**Proposition 4.1.** *Assume that the following conditions are satisfies*

$$(A^{-1})^* PA^{-1} = (B^{-1})^* QB^{-1}, \tag{4.1}$$

$$\sum_{1\le j\le k-1} \binom{k}{j} \Big( \int_a^b p_k u_1^{(j)} \overline{u_1}^{(k-j)}(x)dx$$
$$+ \int_b^c q_k u_2^{(j)} \overline{u_2}^{(k-j)}(x)dx \Big) = 0,\ k = 3, ..., n, \tag{4.2}$$

*for all* $(u_1, u_2) \in D(L_{(n,n)})$. *Then there exists a positive constant* $\rho' > 0$ *such that the operators* $(L - \rho' I)$ *is dissipative.*

**Proof.** We prove that $(L_{n,n} - \rho' I)$ is dissipative for some $\rho' > 0$. The proof requires two steps.

**Step 1.** From (2.1) we have

$$\begin{aligned}\langle L_{(n,n)}u,\ u\rangle_X &= \langle L_{1n}u_1,\ u_1\rangle_{L^2(I_1)} + \langle L_{2n}u_2,\ u_2\rangle_{L^2(I_2)} \\ &= \sum_{0\le k\le n} \int_a^b p_k \overline{u_1} u_1^{(k)}(x)dx + \sum_{0\le k\le n} \int_b^c q_k \overline{u_2} u_2^{(k)}(x)dx.\end{aligned}$$

Therefore we have

$$Re\langle L_{(n,n)}u,\ u\rangle_X = \int_a^b p_0(x)|u_1|^2(x)dx + Re\Big( \int_a^b p_1 \overline{u_1} u_1'(x)dx \Big)$$
$$+ \sum_{2\le k\le n} Re\Big( \int_a^b p_k \overline{u_1} u_1^{(k)}(x)dx \Big) + \int_b^c q_0(x)|u_2|^2(x)dx$$
$$+ Re\Big( \int_b^c q_1 \overline{u_2} u_2'(x)dx \Big) + \sum_{2\le k\le n} Re\Big( \int_b^c q_k \overline{u_2} u_2^{(k)}(x)dx \Big).$$

Form Lemma 4.1 we deduce that

$$Re\langle L_{(n,n)}u,\ u\rangle_X = \frac{1}{2}\Big\{ \int_a^b 2p_0|u_1|^2(x)dx + \sum_{1\le k\le n}(-1)^k\int_a^b p_k^{(k)}|u_1|^2(x)dx$$

$$+\sum_{1\le k\le n}\sum_{0\le j\le k-1} p_k^{(j)}(|u_1|^2)^{(k-1-j)}\big]_a^b - \sum_{2\le k\le n}\sum_{1\le j\le k-1}\binom{k}{j}\int_a^b p_k u_1^{(j)}(\overline{u_1})^{(k-j)}dx$$

$$+\int_b^c 2q_0|u_2|^2(x)dx + \sum_{1\le k\le n}(-1)^k\int_b^c q_k^{(k)}|u_2|^2(x)dx$$

$$+\sum_{1\le k\le n}\sum_{0\le j\le k-1} q_k^{(j)}(|u_2|^2)^{(k-1-j)}\big]_b^c - \sum_{2\le k\le n}\sum_{1\le j\le k-1}\binom{k}{j}\int_b^c q_k u_2^{(j)}(\overline{u_2})^{(k-j)}dx\Big\}$$

$$= \mathcal{R}_1+\mathcal{R}_2+\mathcal{R}_3,$$

where

$$\mathcal{R}_1 = \frac{1}{2}\Big\{\sum_{1\le k\le n}\sum_{0\le j\le k-1} q_k^{(j)}(|u_2|^2)^{(k-1-j)}(c) - \sum_{1\le k\le n}\sum_{0\le j\le k-1} p_k^{(j)}(|u_1|^2)^{(k-1-j)}(a)\Big\},$$

$$\mathcal{R}_2 = \frac{1}{2}\Big\{\sum_{1\le k\le n}\sum_{1\le j\le k} p_k^{(j-1)}(|u_1|^2)^{(k-j)}(b) - \sum_{1\le k\le n}\sum_{1\le j\le k} q_k^{(j-1)}(|u_2|^2)^{(k-j)}(b)\Big\}$$

and

$$\mathcal{R}_3 = \frac{1}{2}\Big\{\int_a^b 2p_0|u_1|^2(x)dx + \sum_{1\le k\le n}(-1)^k\int_a^b p_k^{(k)}|u_1|^2(x)dx$$
$$+\int_b^c 2q_0|u_2|^2(x)dx + \sum_{1\le k\le m}(-1)^k\int_b^c q_k^{(k)}|u_2|^2(x)dx$$
$$-\sum_{2\le k\le n}\sum_{1\le j\le k-1}\binom{k}{j}\int_a^b p_k u_1^{(j)}(\overline{u_1})^{(k-j)}dx$$
$$-\sum_{2\le k\le n}\sum_{1\le j\le k-1}\binom{k}{j}\int_b^c q_k u_2^{(j)}(\overline{u_2})^{(k-j)}dx\Big\}.$$

**Step 2.** Now let us consider each of the above terms separately.
As $(u_1,u_2)\in D(L_{(n,n)})$ we have that

$$\mathcal{R}_1 = \frac{1}{2}\Big\{\sum_{1\le k\le n}\sum_{0\le j\le k-1} q_k^{(j)}\sum_{0\le r\le k-1-j}\binom{k-1-j}{r}\gamma_c^r\gamma_c^{k-j-r}|u_2(c)|^2$$
$$-\sum_{1\le k\le n}\sum_{0\le j\le k-1} p_k^{(j)}\sum_{0\le r\le k-1-j}\binom{k-1-j}{r}\gamma_a^r\gamma_a^{k-1-j-r}|u_1(a)|^2\Big\}.$$

Let

$$A_k(x) = -\frac{1}{2}\Big(\sum_{0\le j\le k-1}\sum_{0\le r\le k-1-j}\binom{k-1-j}{r}\gamma_a^r\gamma_a^{k-1-j-r}p_k^{(j)}(x)\Big)\frac{b-x}{b-a}$$

for $k = 1, 2...., n$, and $x \in I_1$ and

$$B_k(x) = \frac{1}{2}\Bigg(\sum_{0\le j\le k-1}\sum_{0\le r\le k-1-j}\binom{k-1-j}{r}\gamma_c^r\gamma_c^{k-1-j-r}q_k^{(j)}(x)\Bigg)\frac{c-x}{c-b}$$

for $k = 1, 2...., m$, and $x \in I_2$. It follows that

$$\mathcal{R}_1 = \sum_{1\le k\le n}\int_b^c \big(B_k(x)|u_2(x)|^2\big)'dx + \sum_{1\le k\le n}\int_a^b \big(A_k(x)|u_1(x)|^2\big)'dx.$$

Note that for $k = 1, 2, ..., n$

$$\begin{aligned}\big(B_k(x)|u_2(x)|^2\big)' &= B_k'(x)|u_2(x)|^2 + 2B_k(x)Re(\overline{u_2(x)}u_2'(x)) \\ &\le |B_k'(x)||u_2(x)|^2 + 2|B_k(x)||u_2(x)||u_2'(x)|.\end{aligned}$$

Now, not further that for all $\epsilon > 0$, $\epsilon^2|u_2'(x)|^2 + \frac{1}{\epsilon^2}|u_2(x)|^2 \ge 2|u_2(x)||u_2'(x)|$ which implies that $\big(B_k(x)|u_2(x)|^2\big)' \le |B_k'(x)||u_2(x)|^2 + |B_k(x)|\big(\epsilon^2|u_2(x)|^2 + \frac{1}{\epsilon^2}|u_2'(x)|^2\big)$ Similarly for $k = 1.2, ..., n$ $\big(A_k(x)|u_1(x)|^2\big)' \le |A_k'(x)||u_1(x)|^2 + |A_k(x)|\big(\frac{1}{\epsilon^2}|u_1(x)|^2 + \epsilon^2|u_1'(x)|^2.\big)$

$$\begin{aligned}\mathcal{R}_1 \le &\sum_{1\le k\le n}\int_b^c \Big[|B_k'(x)||u_2(x)|^2 + |B_k(x)|\big(\epsilon^2|u_2'(x)|^2 + \frac{1}{\epsilon^2}|u_2(x)|^2\big)\Big]dx \\ &+ \sum_{1\le k\le n}\int_a^b \Big[|A_k'(x)||u_1(x)|^2 + |A_k(x)|\big(\epsilon^2|u_1'(x)|^2 + \frac{1}{\epsilon^2}|u_1(x)|^2\big)\Big]dx.\end{aligned}$$

We then apply our hypotheses (4.1) and (4.2) to check that

$$\begin{aligned}&Re\langle Lu,\ u\rangle_X \le \\ &\frac{1}{2}\Bigg\{\sum_{1\le k\le n}\int_b^c \Big[|B_k'(x)||u_2(x)|^2 + |B_k(x)|\big(\epsilon^2|u_2'(x)|^2 + \frac{1}{\epsilon^2}|u_2(x)|^2\big)\Big]dx \\ &+ \sum_{1\le k\le n}\int_a^b \Big[|A_k'(x)||u_1(x)|^2 + |A_k(x)|\big(\epsilon^2|u_1'(x)|^2 + \frac{1}{\epsilon^2}|u_1(x)|^2\big)\Big]dx\Bigg\} \\ &+ \frac{1}{2}\Big\{\overline{k[u_1]}Pk[u_1](b) - \overline{k[u_2]}Qk[u_2](b)\Big\} \\ &+ \frac{1}{2}\Bigg\{\sum_{1\le k\le n}(-1)^k\int_a^b p_k^{(k)}|u_1|^2(x)dx + \sum_{1\le k\le m}(-1)^k\int_b^c q_k^{(k)}|u_2|^2(x)dx \\ &+ 2\int_a^b p_0|u_1|^2(x)dx - 2\int_a^b p_2|u_1'|^2dx + 2\int_b^c q_0|u_2|^2(x)dx - 2\int_b^c q_2|u_2'|^2dx\Bigg\}.\end{aligned}$$

The above inequality implies

$$
\begin{aligned}
& Re\langle L_{(n,n)}u, u\rangle_X \\
& \leq \frac{1}{2}\Big\{ \int_a^b \Big( \sum_{1\leq k\leq n} \Big( |A'_k(x)| + \frac{|A_k(x)|}{\epsilon^2} + (-1)^k p_k^{(k)}(x) \Big) + 2p_0(x) \Big) |u_1|^2(x)dx \\
& - \int_a^b \Big( 2p_2(x) - \epsilon^2 \sum_{1\leq k\leq n} |A_k(x)|^2 \Big) |u'_1|^2(x)dx \\
& + \int_b^c \Big( \sum_{1\leq k\leq n} \Big( |B'_k(x)| + \frac{|B_k(x)|}{\epsilon^2} + (-1)^k q_k^{(k)}(x) \Big) + 2q_0(x) \Big) |u_2|^2(x)dx \\
& - \int_b^c \Big( 2q_2(x) - \epsilon^2 \sum_{1\leq k\leq m} |B_k(x)|^2 \Big) |u'_2|^2(x)dx \Big\} \\
& + \frac{1}{2}\Big\{ (Ak[u_1])^*(A^{-1})^* P A^{-1} Ak[u_1](b) - (Bk[u_2])^*(B^{-1})^* Q B^{-1} Bk[u_2](b) \Big\}.
\end{aligned}
$$

For sufficiently small $\epsilon > 0$ such that

$$\Big( 2p_2(x) - \epsilon^2 \sum_{1\leq k\leq n} |A_k(x)|^2 \Big) > 0$$

and

$$\Big( 2q_2(x) - \epsilon^2 \sum_{1\leq k\leq n} |B_k(x)|^2 \Big) > 0$$

we obtain that $Re\langle L_{(n,n)}u, u\rangle_X \leq n \max(\alpha, \beta) \|(u_1, u_2)\|_X$ where

$$\alpha = \sup_{a\leq x\leq b} \{ |A'_k(x)| + \frac{|A_k(x)|}{\epsilon^2} + |p_k^{(k)}(x)| + \frac{2}{n}|p_0(x)|,\ k = 1, 2, ..., n\}$$

and

$$\beta = \sup_{b\leq x\leq c} \{ |B'_k(x)| + \frac{|B_k(x)|}{\epsilon^2} + |q_k^{(k)}(x)| + \frac{2}{n}|q_0(x)|\ \ k = 1, 2, ..., n\}.$$

Hence, $\langle \big(L - n\max(\alpha, \beta)\big)u, u\rangle_X \leq 0$. This prove that $(L_{(n,n)} - \rho' I, D(L_{(n,n)}))$ is dissipative. $\square$

**Remark 4.1.** As above, we can define two matrix $\widetilde{P}$ and $\widetilde{Q}$ in $M_n(\mathbb{R})$ satisfying the following equality

$$
\begin{aligned}
& \sum_{1\leq k\leq n} \sum_{0\leq j\leq k-1} \Big( q_k^{(j)} \big(|v_2|^2\big)^{(k-1-j)}(b) + 2(-1)^j \overline{v_2}^{(j)} (q_k v_2)^{(k-1-j)}(b) \Big) \\
& - \Big( \sum_{1\leq k\leq n} \sum_{0\leq j\leq k-1} 2(-1)^j \overline{v_1}^{(j)} (p_k v_1)^{(k-1-j)}(b) + p_k^{(j)} \big(|v_1|^2\big)^{(k-1-j)}(b) \Big) \\
& = \Big\{ \overline{k[v_2](b)} \widetilde{Q} k[v_2](b) - \overline{k[v_1](b)} \widetilde{P} k[v_1](b) \Big\}
\end{aligned}
$$

**Proposition 4.2.** *Assume that the following conditions are satisfies*

$$(A^{-1})^*\widetilde{P}A^{-1}=(B^{-1})^*\widetilde{Q}B^{-1}, \tag{4.3}$$

$$\sum_{1\leq j\leq k-1}\binom{k}{j}\bigg(\int_a^b p_k^{(j)}v_1^{(j)}\overline{v_1}^{(k-j)}(x)dx+\int_b^c q_k^{(j)}v_2^{(j)}\overline{v_2}^{(k-j)}(x)dx\bigg)=0, \quad k=3,...,n, \tag{4.4}$$

*for all* $(v_1,v_2)\in D(L^*_{(n,n)})$. *Then there exists a positive constant* $\rho''>0$ *such that the operators* $(L_{(n,n)}-\rho'' I)^*$ *is dissipative.*

**Proof.** Now to prove that $(L^*_{(n,n)}-\rho'' I, D(L^*))$ is dissipative for some constant $\rho''>0$,we proceed as in the proof of the previous proposition. From Lemma 3.1 it follows that

$$\begin{aligned}\langle L^*_{(n,n)}v,\ v\rangle_X &= \int_a^b p_0|v_1|^2dx-\int_a^b p_1v_1\overline{v_1}'dx+\sum_{2\leq k\leq n}(-1)^k\int_a^b p_kv_1(\overline{v_1})^{(k)}dx\\ &+\sum_{1\leq k\leq n}\sum_{0\leq j\leq k-1}(-1)^j(\overline{v_1})^{(j)}(p_kv_1)^{(k-1-j)}\bigg]_a^b+\int_b^c q_0|v_2|^2dx-\int_b^c q_1v_2\overline{v_2}'dx\\ &+\sum_{2\leq k\leq n}(-1)^k\int_b^c q_kv_2(\overline{v_2})^{(k)}dx+\sum_{1\leq k\leq m}\sum_{0\leq j\leq k-1}(-1)^j(\overline{v_2})^{(j)}(q_kv_2)^{(k-1-j)}\bigg]_b^c\end{aligned}$$

and so that

$$\begin{aligned}&Re\langle L^*_{(n,n)}v,\ v\rangle_X =\\ &\int_a^b p_0|v_1|^2dx-Re\bigg(\int_a^b p_1v_1\overline{v_1}'dx\bigg)+\sum_{2\leq k\leq n}(-1)^kRe\bigg(\int_a^b p_kv_1(\overline{v_1})^{(k)}dx\bigg)\\ &+Re\bigg(\sum_{1\leq k\leq n}\sum_{0\leq j\leq k-1}(-1)^j(\overline{v_1})^{(j)}(p_kv_1)^{(k-1-j)}\bigg]_a^b\bigg)\\ &+\int_b^c q_0|v_2|^2dx-Re\bigg(\int_b^c q_1v_2\overline{v_2}'dx\bigg)\\ &+\sum_{2\leq k\leq n}(-1)^kRe\bigg(\int_b^c q_kv_2(\overline{v_2})^{(k)}dx\bigg)\\ &+Re\bigg(\sum_{1\leq k\leq n}\sum_{0\leq j\leq k-1}(-1)^j(\overline{v_2})^{(j)}(q_kv_2)^{(k-1-j)}\bigg]_b^c\bigg).\end{aligned}$$

Taking into account Lemma 4.1 we get the following equality

$$\begin{aligned}&2Re\langle L^*_{(n,n)}v,\ v\rangle_X =\\ &\int_a^b 2p_0|v_1|^2dx-\bigg(p_1(b)|v_1|^2(b)-p_1(a)|v_1|^2(a)-\int_a^b p_1'|v_1|^2(x)dx\bigg)\end{aligned}$$

$$
\begin{aligned}
&+\sum_{2\le k\le n}\int_a^b p_k^{(k)}|v_1|^2(x)dx\\
&+\sum_{2\le k\le n}(-1)^k\sum_{0\le j\le k-1}\Big(p_k^{(j)}(|v_1|^2)^{(k-1-j)}(b)-p_k^{(j)}(|v_1|^2)^{(k-1-j)}(a)\Big)\\
&-\sum_{2\le k\le n}\sum_{1\le j\le k-1}\binom{k}{j}\int_a^b p_k^{(j)}v_1^{(j)}\overline{v_1}^{(k-j)}(x)dx\\
&+\int_b^c 2q_0|v_2|^2dx-\Big(q_1(c)|v_2|^2(c)-q_1(b)|v_2|^2(b)-\int_b^c q_1'|v_2|^2(x)dx\Big)\\
&+\sum_{2\le k\le m}\int_b^c q_k^{(k)}|v_2|^2(x)dx\\
&+\sum_{2\le k\le m}(-1)^k\sum_{0\le j\le k-1}\Big(q_k^{(j)}(|v_2|^2)^{(k-1-j)}(c)-q_k^{(j)}(|v_2|^2)^{(k-1-j)}(b)\Big)\\
&-\sum_{2\le k\le n}\sum_{1\le j\le k-1}\binom{k}{j}\int_b^c q_k^{(j)}v_2^{(j)}\overline{v_2}^{(k-j)}(x)dx\\
&+2Re\Big(\sum_{1\le k\le n}\sum_{0\le j\le k-1}(-1)^j\overline{v_2}^{(j)}(q_kv_2)^{(k-1-j)}(c)\\
&-\sum_{1\le k\le n}\sum_{0\le j\le k-1}(-1)^j\overline{v_1}^{(j)}(p_kv_1)^{(k-1-j)}(a)\Big)\\
&+2Re\Big(\sum_{1\le k\le n}\sum_{0\le j\le k-1}(-1)^j\overline{v_2}^{(j)}(q_kv_2)^{(k-1-j)}(b)\\
&-\sum_{1\le k\le n}\sum_{0\le j\le k-1}(-1)^j\overline{v_1}^{(j)}(p_kv_1)^{(k-1-j)}(b)\Big).
\end{aligned}
$$

From (4.3) we deduce that

$$
\begin{aligned}
&2Re\langle L^*_{(n,n)}v,\ v\rangle_X\\
&=\Big\{\int_a^b 2p_0|v_1|^2(x)dx+\sum_{1\le k\le n}\int_a^b p_k^{(k)}|v_1|^2(x)dx+\int_b^c 2q_0|v_2|^2(x)dx\\
&+\sum_{1\le k\le n}\int_b^c q_k^{(k)}|v_2|^2(x)dx\Big\}\\
&+\sum_{1\le k\le n}\sum_{0\le j\le k-1}\Big(q_k^{(j)}\big(|v_2|^2\big)^{(k-1-j)}(c)-p_k^{(j)}\big(|v_1|^2\big)^{(k-1-j)}(a)\Big)\\
&+\sum_{1\le k\le n}\sum_{0\le j\le k-1}\Big(q_k^{(j)}\big(|v_2|^2\big)^{(k-1-j)}(b)-p_k^{(j)}\big(|v_1|^2\big)^{(k-1-j)}(b)\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+2Re\Big(\sum_{1\le k\le n}\sum_{0\le j\le k-1}(-1)^j\overline{v_2}^{(j)}(q_kv_2)^{(k-1-j)}(c)\\
&-\sum_{1\le k\le n}\sum_{0\le j\le k-1}(-1)^j\overline{v_1}^{(j)}(p_kv_1)^{(k-1-j)}(a)\Big)\\
&+2Re\Big(\sum_{1\le k\le n}\sum_{0\le j\le k-1}(-1)^j\overline{v_2}^{(j)}(q_kv_2)^{(k-1-j)}(b)\\
&-\sum_{1\le k\le n}\sum_{0\le j\le k-1}(-1)^j\overline{v_1}^{(j)}(p_kv_1)^{(k-1-j)}(b)\Big)\\
&-2\int_a^b p_2'|v_1'|^2(x)dx-2\int_b^c q_2'|v_2'|^2(x)dx=\mathcal{R}_1'+\mathcal{R}_2'+\mathcal{R}_3'
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{R}_1'=&\sum_{1\le k\le n}\sum_{0\le j\le k-1}\Big(q_k^{(j)}\big(|v_2|^2\big)^{(k-1-j)}(c)-p_k^{(j)}\big(|v_1|^2\big)^{(k-1-j)}(a)\Big)\\
&+2Re\Big(\sum_{1\le k\le m}\sum_{0\le j\le k-1}(-1)^j\overline{v_2}^{(j)}(q_kv_2)^{(k-1-j)}(c)\\
&-\sum_{1\le k\le n}\sum_{0\le j\le k-1}(-1)^j\overline{v_1}^{(j)}(p_kv_1)^{(k-1-j)}(a)\Big),
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{R}_2'=&\sum_{1\le k\le n}\sum_{0\le j\le k-1}\Big(q_k^{(j)}\big(|v_2|^2\big)^{(k-1-j)}(b)-p_k^{(j)}\big(|v_1|^2\big)^{(k-1-j)}(b)\Big)\\
&+2Re\Big(\sum_{1\le k\le n}\sum_{0\le j\le k-1}(-1)^j\overline{v_2}^{(j)}(q_kv_2)^{(k-1-j)}(b)\\
&-\sum_{1\le k\le n}\sum_{0\le j\le k-1}(-1)^j\overline{v_1}^{(j)}(p_kv_1)^{(k-1-j)}(b)\Big)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{R}_3'=&\Big\{\int_a^b 2p_0|v_1|^2(x)dx+\sum_{1\le k\le n}\int_a^b p_k^{(k)}|v_1|^2(x)dx+\int_b^c 2q_0|v_2|^2(x)dx\\
&+\sum_{1\le k\le n}\int_b^c q_k^{(k)}|v_2|^2(x)dx\Big\}-2\int_a^b p_2'|v_1'|^2(x)dx-2\int_b^c q_2'|v_2'|^2(x)dx.
\end{aligned}
$$

Now let us consider each of the term in the above equation separately: From easy calculations, it follows that for $(v_1,v_2)\in D(L_{n,n}^*)$

$$
\mathcal{R}_1'=\sum_{1\le k\le n}\sum_{0\le j\le k-1}\sum_{0\le r\le k-1-j}\binom{k-1-j}{r}
$$

$$\cdot \Big( \omega_c^r \omega_c^{k-1-j-r} q_k^{(j)}(c) + 2(-1)^j \omega_c^j \omega_c^{k-1-j-r} q_k^{(r)}(c) \Big) |v_2(c)|^2$$

$$- \sum_{1 \le k \le n} \sum_{0 \le j \le k-1} \sum_{0 \le r \le k-1-j} \binom{k-1-j}{r}$$

$$\cdot \Big( \omega_a^r \omega_a^{k-1-j-r} p_k^{(j)}(a) + 2(-1)^j \omega_a^j \omega_a^{k-1-j-r} p_k^{(r)}(a) \Big) |v_1(a)|^2.$$

Put for $k = 1, 2, ..., n$

$$C_k(x) = \sum_{0 \le j \le k-1} \sum_{0 \le r \le k-1-j} \binom{k-1-j}{r}$$

$$\cdot \Big( \omega_a^r \omega_a^{k-1-j-r} p_k^{(j)}(a) + 2(-1)^j \omega_a^j \omega_a^{k-1-j-r} p_k^{(r)}(a) \Big) \frac{b-x}{b-a},\ x \in I_1,$$

and

$$D_k(x) = \sum_{0 \le j \le k-1} \sum_{0 \le r \le k-1-j} \binom{k-1-j}{r}$$

$$\cdot \Big( \omega_c^r \omega_c^{k-1-j-r} q_k^{(j)}(c) + 2(-1)^j \omega_c^j \omega_c^{k-1-j-r} q_k^{(r)}(c) \Big) \frac{c-x}{c-b},\ x \in I_2.$$

From this we conclude

$$\mathcal{R}_1' = \sum_{1 \le k \le n} \Big[ \int_a^b \Big( C_k(x) |v_1(x)|^2 \Big)' dx + \int_b^c \Big( D_k(x) |v^2(x)|^2 \Big)' dx \Big].$$

In same way as in proof of Proposition 4.1,it is possible to prove that

$$\begin{aligned} \mathcal{R}_1' \quad \le \quad & \sum_{1 \le k \le n} \Big\{ \int_b^c \Big[ |D_k'(x)||v_2(x)|^2 + |D_k(x)| \big( \epsilon^2 |v_2'(x)|^2 + \frac{1}{\epsilon^2} |v_2(x)|^2 \big) \Big] dx \\ & + \int_a^b \Big[ |C_k'(x)||v_1(x)|^2 + |C_k(x)| \big( \epsilon^2 |v_1'(x)|^2 + \frac{1}{\epsilon^2} |v_1(x)|^2 \big) \Big] dx \Big\}. \end{aligned}$$

Concerning $\mathcal{R}_2'$ we have the following identity

$$\mathcal{R}_2' = Re \Big\{ \overline{k[v_2](b)} \widetilde{Q} k[v_2](b) - \overline{k[v_1](b)} \widetilde{P} k[v_1](b) \Big\}.$$

According to the interface condition and equation (3.4) it is possible to prove that

$$\begin{aligned} \mathcal{R}_2' &= Re \Big\{ \overline{k[v_2](b)} \widetilde{Q} k[v_2](b) - \overline{k[v_1](b)} \widetilde{P} k[v_1](b) \Big\} \\ &= Re \Big\{ \big( B k[v_2](b) \big)^* \big( B^{-1} \big)^* \widetilde{Q} B^{-1} B k[v_2](b) \\ &- \big( A k[v_1](b) \big)^* \big( A^{-1} \big)^* \widetilde{P} A^{-1} A k[v_1](b) \Big\} = 0. \end{aligned}$$

From the above calculations, we get the following estimation of $Re\,\langle L^*v \mid v\rangle_X$ as

$$\begin{aligned}
Re\langle L^*v, v\rangle_X \leq \frac{1}{2}\Big\{ &\int_a^b \Big( \sum_{1\leq k\leq n} \Big( |C_k'(x)| + \frac{|C_k(x)|}{\epsilon^2} + p_k^{(k)}(x) \Big) + 2p_0(x) \Big) |v_1|^2(x)dx \\
&- \int_a^b \Big( 2p_2'(x) - \epsilon^2 \sum_{1\leq k\leq n} |C_k(x)|^2 \Big) |v_1'|^2(x)dx \\
&+ \int_b^c \Big( \sum_{1\leq k\leq n} \Big( |D_k'(x)| + \frac{|D_k(x)|}{\epsilon^2} + q_k^{(k)}(x) \Big) + 2q_0(x) \Big) |v_2|^2(x)dx \\
&- \int_b^c \Big( 2q_2'(x) - \epsilon^2 \sum_{1\leq k\leq n} |D_k(x)|^2 \Big) |v_2'|^2(x)dx \Big\}.
\end{aligned}$$

For sufficiently small $\epsilon > 0$ such that

$$\Big( 2p_2'(x) - \epsilon^2 \sum_{1\leq k\leq n} |C_k(x)|^2 \Big) > 0$$

and

$$\Big( 2q_2'(x) - \epsilon^2 \sum_{1\leq k\leq m} |D_k(x)|^2 \Big) > 0$$

we obtain that $Re\langle L^*v, v\rangle_X \leq \max(\alpha', \beta')\|v\|_X$ where

$$\alpha' = \sup_{a\leq x\leq b} \{ |C_k'(x)| + \frac{|C_k(x)|}{\epsilon^2} + |p_k^{(k)}(x)| + \frac{2}{n}|p_0(x)|,\ k = 1, 2, ..., n\}$$

and

$$\beta' = \sup_{b\leq x\leq c} \{ |D_k'(x)| + \frac{|D_k(x)|}{\epsilon^2} + |q_k^{(k)}(x)| + \frac{2}{n}|q_0(x)|\ \ k = 1, 2, ..., m\}.$$

Hence,

$$Re\Big\langle \big(L^* - \underbrace{n\max(\alpha', \beta')}_{\rho''}\big)v, v \Big\rangle_X \leq 0.$$

□

**Theorem 4.1.** *Assume that the conditions* (4.1) − (4.2) − (4.3) − (4.4) *are satisfies. Then there exists a positive constant* $\gamma > 0$ *such that the operators* $(L - \gamma I)$ *is m-dissipative.*

**Proof.** From Proposition 4.1 it follows that there exists a constant $\rho' > 0$ such that $Re\langle L_{(n,n)}u,\ u\rangle \leq \rho'\|u\|_X^2$ For all $u \in D(L_{n,n})$. Similarly, by Proposition 4.2 we have for all $v \in D(L_{n,n}^*)$ $Re\langle L_{(n,n)}^*v,\ v\rangle \leq \rho''\|v\|_X^2$ for some $\gamma_0' > 0$. If we put $\gamma = \max(\rho', \rho'')$, we can conclude that $\big(L_{(n,n)} - \gamma I\big)$ and $\big(L_{(n,n)} - \gamma I\big)^*$ are dissipative. □

The following corollary is an immediate consequence of Theorem 4.1.

**Corollary 4.1.** *Under the assumptions of Theorem 4.1, the operator* $(L_{(n,n)}, D(L_{n,n}))$ *generates a contraction semigroup on* $X$.

## 5. Analyticity of the semigroup generated by $(L, D(L))$

In this section we shall apply the general results of Section 4. In order to establish the analyticity of the semigroup generated by $(L_{(n,n)}, D(L_{(n,n)}))$, we need the following theorem.

**Theorem 5.1** ([6]). *Let* $A$ *be a densely defined operator in a complex Hilbert space* $X$ *such that; for* $u \in D(A)$, $Re\,\langle Au\,,u\rangle \pm \delta Im\,\langle Au\,,u\rangle \leq 0$. *Then* $A$ *generates an analytic semigroup of contractions.*

In analogy with [3, Theorem 2.4] and [10, Theorem 4] we prove the following result.

**Theorem 5.2.** *Assume that* (4.1), (4.2), (4.3), (4.4) *are hold and suppose in addition that there exist real numbers* $\mu > 0$ *and* $\nu > 0$ *such that*

$$\sum_{2\leq k\leq n}\left(\int_a^b |p_k(x)||u_1^{(k)}(x)|^2dx + \int_b^c |q_k(x)||u_2^{(k)}(x)|^2dx\right.$$
$$\leq \mu\int_a^b |u_1'(x)|^2dx + \nu\int_b^c |u_2'(x)|^2dx. \tag{5.1}$$

*Then the operator* $L_{(n,n)}$ *generates an analytic semigroup.*

**Proof.**

$$\left\langle L_{(n,n)}u,\ u\right\rangle_X = \sum_{0\leq k\leq n}\left(\int_a^b p_k\overline{u_1}u_1^{(k)}(x)dx + \int_b^c q_k\overline{u_2}u_2^{(k)}(x)dx\right)$$
$$= \int_a^b p_0|u_1|^2(x)dx + \int_b^c q_0|u_2|^2(x)dx + \int_a^b p_1\overline{u_1}u_1'(x)dx + \int_b^c q_1\overline{u_2}u_2'(x)dx$$
$$+ \sum_{2\leq k\leq n}\left(\int_a^b p_k\overline{u_1}u_1^{(k)}(x)dx + \int_b^c q_k\overline{u_2}u_2^{(k)}(x)dx\right)$$

We deduce that

$$Im\left\langle L_{(n,n)}u,\ u\right\rangle_X = \int_a^b p_1 Im\left(\overline{u_1}u_1'(x)\right)dx + \int_b^c q_1 Im\left(\overline{u_2}u_2'(x)\right)dx$$
$$+ \sum_{2\leq k\leq n}\left(\int_a^b p_k Im\left(\overline{u_1}u_1^{(k)}(x)\right)dx + \int_b^c q_k Im\left(\overline{u_2}u_2^{(k)}(x)\right)dx.\right.$$

Hence we can conclude that

$$
\begin{aligned}
&|Im\left\langle L_{(n,n)}u,\ u\right\rangle_X| \le \int_a^b |p_1||\overline{u_1}||u_1'(x)|dx + \int_b^c |q_1||\overline{u_2}||u_2'(x)|dx\\
&+\sum_{2\le k\le n}\left(\int_a^b |p_k||\overline{u_1}||u_1^{(k)}|(x)dx + \int_b^c |q_k||\overline{u_2}||u_2^{(k)}|(x)dx\right.\\
&\le \int_a^b |p_1|\big(\epsilon^{-2}|u_1(x)|^2+\epsilon^2|u_1'(x)|^2\big)dx + \int_b^c |q_1|\big(\epsilon^{-2}|u_2(x)|^2+\epsilon^2|u_2'(x)|^2\big)dx\\
&+\sum_{2\le k\le n}\left(\int_a^b |p_k|\epsilon^{-2}|u_1(x)|^2dx + \int_b^c |q_k|\epsilon^{-2}|u_2(x)|^2dx\right)\\
&+\sum_{2\le k\le n}\left(\int_a^b |p_k(x)|\epsilon^{2}|u_1^{(k)}(x)|^2dx + \int_b^c |q_k(x)|\epsilon^{2}|u_2^{(k)}(x)|^2dx\right)\\
&=\int_a^b\left(\epsilon^{-2}\sum_{1\le k\le n}|p_k(x)|\right)|u_1(x)|^2dx+\int_a^b\epsilon^2|p_1(x)||u_1'(x)|^2dx\\
&+\int_b^c\left(\epsilon^{-2}\sum_{1\le k\le n}|q_k(x)|\right)|u_2(x)|^2dx+\int_b^c\epsilon^2|q_1(x)||u_2'(x)|^2dx\\
&+\sum_{2\le k\le n}\left(\int_a^b |p_k(x)|\epsilon^{2}|u_1^{(k)}(x)|^2dx + \int_b^c |q_k(x)|\epsilon^{2}|u_2^{(k)}(x)|^2dx\right).
\end{aligned}
$$

From (5.1) it follows that

$$
\begin{aligned}
&|Im\left\langle L_{(n,n)},\ u\right\rangle_X|\\
&\le \int_a^b\left(\epsilon^{-2}\sum_{1\le k\le n}|p_k(x)|\right)|u_1(x)|^2dx+\int_a^b\left(\mu+\epsilon^2|p_1(x)|\right)|u_1'(x)|^2dx\\
&+\int_b^c\left(\epsilon^{-2}\sum_{1\le k\le n}|q_k(x)|\right)|u_2(x)|^2dx+\int_b^c\left(\nu+\epsilon^2|q_1(x)|\right)|u_2'(x)|^2dx.
\end{aligned}
$$

Then we have for a sufficiently small $\epsilon > 0$

$$
\begin{aligned}
&Re\left\langle L_{n,n}u,\ u\right\rangle_X+\delta|Im\left\langle L_{n,n}u,\ u\right\rangle_X|\\
&\le\frac{1}{2}\Bigg\{\int_a^b\Bigg(\sum_{1\le k\le n}\Bigg(|A_k'(x)|+\frac{|A_k(x)|}{\epsilon^2}\\
&+(-1)^kp_k^{(k)}(x)+\frac{2\delta}{\epsilon^2}|p_k(x)|\Bigg)+2p_0(x)\Bigg)|u_1|^2(x)dx\\
&-\int_a^b\left(2p_2(x)-\epsilon^2\sum_{1\le k\le n}|A_k(x)|^2-2\delta\mu-2\delta\epsilon^2|p_1(x)|\right)|u_1'|^2(x)dx
\end{aligned}
$$

$$
\begin{aligned}
&+\int_b^c \Bigg( \sum_{1\le k\le n} \Bigg( |B_k'(x)| + \frac{|B_k(x)|}{\epsilon^2} \\
&+ (-1)^k q_k^{(k)}(x) + \frac{2\delta}{\epsilon^2}|q_k(x)| \Bigg) + 2q_0(x) \Bigg) |u_2|^2(x)dx \\
&- \int_b^c \Bigg( 2q_2(x) - \epsilon^2 \sum_{1\le k\le n} |B_k(x)|^2 - 2\delta\nu - 2\epsilon^2\delta|q_1(x)| \Bigg) |u_2'|^2(x)dx \Bigg\} \\
&+ \frac{1}{2}\Big\{ (Ak[u_1])^*(A^{-1})^* P A^{-1} Ak[u_1](b) - (Bk[u_2])^*(B^{-1})^* Q B^{-1} Bk[u_2](b) \Big\} \\
&\le \rho \|u\|_X^2
\end{aligned}
$$

where

$$
\rho = \max_{1\le k\le n} \Bigg( n \sup_{a\le x\le b} (\alpha_k(x),\ n \sup_{b\le x\le c} (\beta_k(x)) \Bigg),
$$

where

$$
\alpha_k(x) = |A_k'(x)| + \frac{|A_k(x)|}{\epsilon^2} + |p_k^{(k)}(x)| + \frac{2\delta}{\epsilon^2}|p_k(x)| + \frac{2}{n}|p_0(x)|,\ k = 1, 2, ..., n
$$

and

$$
\beta_k(x) = |B_k'(x)| + \frac{|B_k(x)|}{\epsilon^2} + (-1)^k q_k^{(k)}(x) + \frac{2\delta}{\epsilon^2}|q_k(x)| + \frac{2}{n}|q_0(x)|, \quad k = 1, 2, ..., n.
$$

The proof of this theorem is completed. □

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